

THE BUBBLE SKEIN ELEMENT AND APPLICATIONS

MUSTAFA HAJIJ

ABSTRACT. We study a certain skein element in the relative Kauffman bracket skein module of the disk with some marked points, and expand this element in terms linearly independent elements of this module. This expansion is used to study the head and the tail of the colored Jones polynomial and to obtain an easy determination of the theta coefficients.

1. INTRODUCTION

In [2] Armond and Dasbach introduced the head and the tail of the colored Jones Polynomial $J_{N,L}$ of an alternating knot L , two link invariants which take the form of infinite power series with integer coefficients. Skein theoretic techniques have been used in [1] and [2] to understand the head and tail of an alternating link. Write $\mathcal{S}(S^3)$ to denote the Kauffman Bracket Skein Module of S^3 . Let L be an alternating link and let D be a reduced link diagram of L . Write $S_B(D)$ to denote the all- B smoothing state of D , the state obtained by replacing each crossing by a B smoothing. Let $S_B^{(n)}(D)$ be the skein element obtained from $S_B(D)$ by decorating each circle in this state with the n^{th} Jones-Wenzl idempotent and replacing each place where we had a crossing in D with the $(2n)^{th}$ projector. It was proven in [1] that for an adequate link L the first N coefficients of $J_{N,L}$ coincide with the first N coefficients of the skein element in $S_B^{(n)}(D)$. Our work here initially aimed to understand $S_B^{(n)}(D)$ for an alternating link diagram D . Initial examinations of various examples of $S_B^{(n)}(D)$ showed that a certain skein element in the relative Kauffman bracket skein module of the disk with some marked points mostly shows up as sub-skein element of $S_B^{(n)}(D)$. We will call this element the bubble skein element. We eventually found various applications for the bubble element. In fact, nearly all fundamental skein theoretic identities can be derived from the bubble skein formula and one other simple skein identity [4].

1.1. Plan of the paper. In section 2, we review the skein theory basics needed in this paper. In section 3, we give a recursive formula for the bubble skein element. In section 4, we use the recursive identity for the bubble skein element to expand this element in terms of a linearly independent set in the relative module of the disk. Finally, in section 5, we use our results to give an easy way to determine a theta graph spin network evaluation in $\mathcal{S}(S^3)$ and study the element $S_B^{(n)}$ for a family of knots.

1.2. Acknowledgements. I would like to thank Oliver Dasbach whose guidance made this possible. I would also like to thank Pat Gilmer for teaching me skein theory and for his helpful discussions, Cody Armond for many valuable conversations, Khaled Bataineh and Kyle Istvan for reading early drafts carefully and pointing out many corrections.

2. SKEIN THEORY

In this section we review the fundamentals of the Kauffman Bracket Skein Modules and introduce the skein modules that will be used for our purpose. Furthermore, we discuss the recursive definition of Jones-Wenzl idempotent and recall some of its basic properties. For more details about linear skein theory associated with the Kauffman Bracket, see [8], [11], and [14].

Definition 2.1. (J. Przytycki [14]) Let M be an oriented 3-manifold. Let R be a commutative ring with identity and a fixed invertable element A . The Kauffman Bracket Skein Module $\mathcal{S}(M; R, A)$ is the free module generated by the set of isotopy classes of framed links in M (including the empty knot) modulo the submodule generated by the Kauffman relations [7]:

$$\left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) - A \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) = -A^{-1} \left(\begin{array}{c} \cup \\ \cap \end{array} \right), \quad L \sqcup \bigcirc + (A^2 + A^{-2})L$$

A relative version of the Kauffman bracket skein module can be defined when M has a boundary. We specify a finite (possibly empty) set of points on the boundary of M . The relative module is the free module generated by the set of isotopy classes of framed links together with arcs joining the designated boundary points of M modulo the Kauffman relations specified above.

When the manifold M is homeomorphic to $F \times I$, where F is an oriented planer surface with a finite set of points (possibly empty) in its boundary ∂F , one could define the (relative) Kauffman bracket skein module of F . In this case one considers an appropriate version of link diagrams in F instead of framed links in M . The isomorphism between $\mathcal{S}(M; R, A)$ and $\mathcal{S}(F; R, A)$ that sends a framed link to its link diagram will be used to identify these two skein modules.

In this paper R will be $\mathbb{Q}(A)$, the field generated by the indeterminate A over the rational numbers. We will work with the Kauffman bracket skein module of the 3-sphere $\mathcal{S}(S^3; R, A)$. Let D be any diagram in S^2 . Using the definition of the normalized Kauffman bracket, we can write $D = \langle D \rangle \phi$ in $\mathcal{S}(S^3; R, A)$. This provides an isomorphism $\mathcal{S}(S^3; R, A) \cong \mathbb{Q}(A)$, induced by sending D to $\langle D \rangle$. We will also work with the relative skein module $\mathcal{S}(D^2; R, A, 2n)$, where the rectangular disk D^2 has n designated points on the top edge and n designated points on the bottom edge. This relative skein module can be thought of as the $Q(A)$ -module generated by all (n, n) -tangle diagrams in D^2 modulo the Kauffman relations. The relative skein module $\mathcal{S}(D^2; R, A, 2n)$ admits a multiplication given by juxtaposition of two diagrams in $\mathcal{S}(D^2; R, A, 2n)$. With this multiplication $\mathcal{S}(D^2; R, A, 2n)$ is an associative algebra over $Q(A)$ known as the n^{th} Temperley-Lieb algebra TL_n . For each n there exists an idempotent $f^{(n)}$ in TL_n that plays a central role in the Witten-Reshetikhin-Turaev Invariants for $SU(2)$. See [8], [10], and [16]. The idempotent $f^{(n)}$, known as the n^{th} Jones-Wenzl idempotent, was discovered by Jones [5] and it has a recursive formula due to Wenzl [17]. We will use this recursive formula to define $f^{(n)}$. Further, we will adapt a graphical notation for $f^{(n)}$ which is due Lickorish [10]. In this graphical notation one thinks of $f^{(n)}$ as an empty box with n strands entering and n strands leaving the opposite side. *The Jones-Wenzl idempotent* is defined by:

$$\begin{array}{c} \text{Diagram 1: } n \text{ strands entering a box from the top, } n-1 \text{ strands exiting from the bottom, and } 1 \text{ strand exiting from the bottom.} \\ \text{Diagram 2: } n-1 \text{ strands entering a box from the top, } n-2 \text{ strands exiting from the bottom, and } 1 \text{ strand exiting from the bottom.} \\ \text{Diagram 3: } n-1 \text{ strands entering a box from the top, } n-2 \text{ strands exiting from the bottom, and } 1 \text{ strand exiting from the bottom.} \\ \text{Diagram 4: } 1 \text{ strand entering a box from the top, } 1 \text{ strand exiting from the bottom, and } n-1 \text{ strands exiting from the bottom.} \end{array} \quad (2.1)$$

where

$$\Delta_n = (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}.$$

The polynomial Δ_n is related to $[n+1]$, the $(n+1)^{\text{th}}$ quantum integer, by $\Delta_n = (-1)^n [n+1]$. It is common to use the substitution $a = A^2$ when one works with quantum integers.

We assume that $f^{(0)}$ is the empty diagram. The Jones-Wenzl idempotent satisfies

$$\Delta_n = \text{circle with } n \text{ strands} , \quad \begin{array}{c} m \\ | \\ \text{box} \\ | \\ m+n \end{array} = \begin{array}{c} n \\ | \\ \text{box} \\ | \\ m+n \end{array} , \quad \begin{array}{c} m \quad n \\ | \quad | \\ \text{box} \\ | \\ m+n+2 \end{array} = 0 \quad (2.2)$$

and

$$\begin{array}{c} m \\ | \\ \text{circle} \\ | \\ m+n \end{array} = \frac{\Delta_{m+n}}{\Delta_n} \begin{array}{c} n \\ | \\ \text{box} \\ | \\ n \end{array} \quad (2.3)$$

The last relative Kauffman bracket skein module that we will need here is a certain submodule of the relative skein module of D^2 with $m + n + m' + n'$ specified points on its boundary. To define this submodule we partition this set of points into four sets of m , n , m' , and n' points, respectively, and at each cluster of these points we place an appropriate idempotent, i.e., the one whose color matches the cardinality of this cluster. See Figure 1. We will denote this skein module by $\mathcal{D}_{m',n'}^{m,n}$.

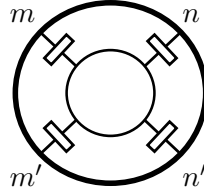
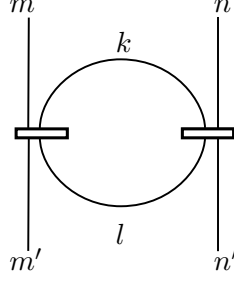


FIGURE 1. The relative skein module $\mathcal{D}_{m',n'}^{m,n}$

We assume that a diagram is zero if it has a strand labeled by a negative number.

3. A RECURSIVE FORMULA FOR THE BUBBLE SKEIN ELEMENT

In this section we use the recursive definition of the Jones-Wenzl idempotent to obtain a recursive formula for a certain skein element, denoted by $\mathcal{B}_{m',n'}^{m,n}(k,l)$, in the module $\mathcal{D}_{m',n'}^{m,n}$. See Figure 2. We will call such an element in $\mathcal{D}_{m',n'}^{m,n}$ a *bubble skein element*. To obtain a recursive formula for $\mathcal{B}_{m',n'}^{m,n}(k,l)$ we start by expanding $\mathcal{B}_{m',n'}^{m,n}(k,1)$ in Lemma 3.1 and then use this expansion to obtain a recursion for an arbitrary bubble skein element in Lemma 3.4. The recursive formula of $\mathcal{B}_{m',n'}^{m,n}(k,l)$ obtained in this section will be used in Theorem 4.1 to write a bubble skein element as a $\mathbb{Q}(A)$ -linear sum of linearly independent elements in $\mathcal{D}_{m',n'}^{m,n}$.

FIGURE 2. The bubble skein element $\mathcal{B}_{m',n'}^{m,n}(k,l)$

We denote the rational functions $\frac{\Delta_{n+k}\Delta_{m+k-1}-\Delta_n\Delta_{m-1}}{\Delta_{n+k-1}\Delta_{m+k-1}}$ and $\frac{\Delta_{m-1}\Delta_{n-1}}{\Delta_{n+k-1}\Delta_{m+k-1}}$ by $\alpha_{m,n}^k$ and $\beta_{m,n}^k$, respectively.

Lemma 3.1. Let $m, n, m', n' \geq 0$ and $k \geq 1$. Then we have

$$\mathcal{B}_{m',n'}^{m,n}(k,1) = \alpha_{m,n}^k \mathcal{B}_{m',n'}^{m,n}(k-1,1) + \beta_{m,n}^k \mathcal{B}_{m',n'}^{m,n}(1,k)$$

Proof. First we consider the trivial cases when one of the integers m, n, m', n' is zero. Observe that $m+k=1+m'$ and $n+k=1+n'$. Since $k \geq 1$, we know that $m \leq m'$ and $n \leq n'$. If $m' = n' = 0$, then this implies that m and n must be zero and k is 1. Hence $\mathcal{B}_{0,0}^{0,0}(1,1) = \alpha_{0,0}^1 = \Delta_1$ and we are done. Here we used our convention that a diagram is zero if it has a strand colored by a negative number. If $\min(m, n) = 0$ or $\min(m', n') = 0$, then the result follows from (2.3).

When $\min(m, n) \neq 0$, we use induction on k . For $k = 1$ we apply the recursive definition of the Jones-Wenzel idempotent (2.1) on the projector $f^{(m+1)}$ that appears in $\mathcal{B}_{m',n'}^{m,n}(1,1)$ to obtain

$$\mathcal{B}_{m',n'}^{m,n}(1,1) = \text{Diagram with single bars} - \frac{\Delta_{m-1}}{\Delta_m} \text{Diagram with crossings} \quad (3.1)$$

Using identity (2.3) in the first term and expanding the projector $f^{(n+1)}$ the second term in (3.1) implies

$$\mathcal{B}_{m',n'}^{m,n}(1,1) = \frac{\Delta_{n+1}\Delta_m - \Delta_n\Delta_{m-1}}{\Delta_n\Delta_m} \text{Diagram with single bar} + \frac{\Delta_{m-1}\Delta_{n-1}}{\Delta_n\Delta_m} \text{Diagram with crossings} \quad (3.2)$$

$$\begin{aligned}
& \text{Diagram 1} = \frac{\Delta_{n+k}\Delta_{m+k-1} - \Delta_{n+k-1}\Delta_{m+k-2}}{\Delta_{m+k-1}\Delta_{n+k-1}} \text{Diagram 2} \\
& + \frac{\Delta_{m+k-2}\Delta_{n+k-2}}{\Delta_{m+k-1}\Delta_{n+k-1}} \left(\frac{\Delta_{n+k-1}\Delta_{m+k-2} - \Delta_n\Delta_{m-1}}{\Delta_{m+k-2}\Delta_{n+k-2}} \text{Diagram 3} \right. \\
& \left. + \frac{\Delta_{m-1}\Delta_{n-1}}{\Delta_{m+k-2}\Delta_{n+k-2}} \text{Diagram 4} \right)
\end{aligned}$$

Collecting similar terms together, we obtain

$$\begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \begin{array}{c} k \\ \bigcirc \\ 1 \end{array} = \frac{\Delta_{n+k} \Delta_{m+k-1} - \Delta_n \Delta_{m-1}}{\Delta_{n+k-1} \Delta_{m+k-1}} \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \begin{array}{c} k-1 \\ \bigcirc \\ k-1 \end{array} + \frac{\Delta_{m-1} \Delta_{n-1}}{\Delta_{n+k-1} \Delta_{m+k-1}} \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \begin{array}{c} 1 \\ \bigcirc \\ k \end{array}$$

□

Remark 3.2. Note that, while the symmetry with respect to the variables m and n in the function $\beta_{m,n}^k$ is clear, the rational function $\alpha_{m,n}^k$ appears to be asymmetric with respect to these variables. It is routine to verify that $\Delta_{n+k} \Delta_{m+k-1} - \Delta_n \Delta_{m-1} = \Delta_{m+k} \Delta_{n+k-1} - \Delta_m \Delta_{n-1}$ and hence $\alpha_{m,n}^k$ is also symmetric with respect to the variables m and n . This also could be seen to follow from the proof of Lemma 3.1. To see this note that in Lemma 3.1 we obtain (3.4) by expanding $f^{(m+k)}$ first and $f^{(n+k)}$ second. Further, the coefficients $\alpha_{m,n}^k$ and $\beta_{m,n}^k$ are a result of an iterative application of (3.4). We could have done the expansion of the bubble in the opposite order, by expanding $f^{(n+k)}$ first and $f^{(m+k)}$ second, and obtain an identity that is identical to (3.4) except m and n are swapped. An iteration of this identity would yield the same coefficient $\beta_{m,n}^k$ and forces $\alpha_{m,n}^k$ to be the same as $\alpha_{n,m}^k$.

We will need the following identity. A proof can be found in [8] and [12].

Lemma 3.3. $\Delta_{m+k} \Delta_{n+k-1} - \Delta_m \Delta_{n-1} = \Delta_{m+n+k} \Delta_{k-1}$

Note that the previous identity can be used to obtain a more manageable formula for $\alpha_{m,n}^k$.

Lemma 3.4. Let $m, n, m', n' \geq 0$, $k, l \geq 1$ and $k \geq l$. Then we have

$$\begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \begin{array}{c} k \\ \bigcirc \\ l \end{array} = \alpha_{m,n}^k \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \begin{array}{c} k-1 \\ \bigcirc \\ l-1 \end{array} + \beta_{m,n}^k \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \begin{array}{c} 1 \\ \bigcirc \\ k \end{array} \quad (3.5)$$

Proof. We start again by considering the trivial cases when one of the integers m, n, m', n' is zero. Note that, as in Lemma 3.1, $m \leq m'$ and $n \leq n'$. If $m' = n' = 0$, then $\mathcal{B}_{0,0}^{0,0}(k, k) = \alpha_{0,0}^k = \Delta_k$. If $\min(m, n) = 0$ or $\min(m', n') = 0$, then the result follows from (2.3). Now suppose that $\min(m, n) \neq 0$ and apply the recursive definition of the Jones-Wenzel idempotent on the projector $f^{(m+k)}$ that appears in $\mathcal{B}_{m',n'}^{m,n}(k, l)$

$$\begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \begin{array}{c} k \\ \bigcirc \\ l \end{array} = \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \begin{array}{c} k-1 \\ \bigcirc \\ l-1 \end{array} - \frac{\Delta_{m+k-2}}{\Delta_{m+k-1}} \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \begin{array}{c} k-1 \\ \bigcirc \\ l-1 \end{array} \quad (3.6)$$

$$\begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} k \\ \bigcirc \\ l \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} = \frac{\Delta_{n+k} \Delta_{m+k-1} - \Delta_{n+k-1} \Delta_{m+k-2}}{\Delta_{n+k-1} \Delta_{m+k-1}} \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} k-1 \\ \bigcirc \\ l-1 \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} + \frac{\Delta_{m+k-2} \Delta_{n+k-2}}{\Delta_{n+k-1} \Delta_{m+k-1}} \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} k-1 \\ \bigcirc \\ 1 \\ \bigcirc \\ l-1 \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \quad (3.7)$$
$$\begin{aligned}
& \text{Diagram 1} = \frac{\Delta_{n+k}\Delta_{m+k-1} - \Delta_{n+k-1}\Delta_{m+k-2}}{\Delta_{n+k-1}\Delta_{m+k-1}} \text{Diagram 2} \\
& + \frac{\Delta_{m+k-2}\Delta_{n+k-2}}{\Delta_{n+k-1}\Delta_{m+k-1}} \left(\frac{\Delta_{n+k-1}\Delta_{m+k-2} - \Delta_n\Delta_{m-1}}{\Delta_{n+k-2}\Delta_{m+k-2}} \text{Diagram 3} \right. \\
& \left. + \frac{\Delta_{m-1}\Delta_{n-1}}{\Delta_{n+k-2}\Delta_{m+k-2}} \text{Diagram 4} \right) \\
& = \frac{\Delta_{n+k}\Delta_{m+k-1} - \Delta_n\Delta_{m-1}}{\Delta_{n+k-1}\Delta_{m+k-1}} \text{Diagram 5} + \frac{\Delta_{m-1}\Delta_{n-1}}{\Delta_{n+k-1}\Delta_{m+k-1}} \text{Diagram 6}.
\end{aligned}$$
☐

In this section we will use the recursive formula we obtained in Lemma 3.4 for $\mathcal{B}_{m',n'}^{m,n}(k,l)$ to expand this element as a $\mathbb{Q}(A)$ -linear sum of certain linearly independent skein elements in $\mathcal{D}_{m',n'}^{m,n}$. Then we will use Theorem 4.1 together with Lemma 3.4 to determine a recursive formula for the

$$\begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \quad \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \quad \begin{array}{c} k \\ \text{---} \text{---} \text{---} \end{array} \quad \begin{array}{c} l \\ \text{---} \text{---} \text{---} \end{array} = \sum_{i=0}^{\min(m,n,l)} \left[\begin{array}{cc} m & n \\ k & l \end{array} \right]_i \quad \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \quad \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \quad \begin{array}{c} i \\ \text{---} \text{---} \end{array} \quad \begin{array}{c} k-l+i \\ \text{---} \text{---} \end{array} \quad (4.1)$$
$$\begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} k \\ \bigcirc \\ l \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} = \sum_{i=0}^{\min(m', n', k)} \left[\begin{array}{cc} n' & m' \\ k & l \end{array} \right]_i \begin{array}{c} m \\ | \\ \text{---} \\ | \\ m' \end{array} \begin{array}{c} l-k+i \\ \bigcirc \\ i \end{array} \begin{array}{c} n \\ | \\ \text{---} \\ | \\ n' \end{array} \quad (4.2)$$

where $s = \min(m, n, l)$. Now the result follows by noticing that the variables m, n, m', n', k and l are related by the equations $m + k = m' + l$ and $n + k = n' + l$ and hence it is sufficient to index

the coefficients in the expansion of $\mathcal{B}_{m',n'}^{m,n}(k,l)$ in terms of the previous linearly independent set by five indices. (2) Follows from (1). \square

Now we determine the coefficients $\begin{bmatrix} m & n \\ k & l \end{bmatrix}_i$.

Proposition 4.2. Let $m, n, m', n' \geq 0$, $\mathcal{B}_{m',n'}^{m,n}(k,l)$ be a bubble skein element in $\mathcal{D}_{m',n'}^{m,n}$ such that $k \geq l$. Then, for $0 \leq i \leq \min(m, n, l)$, the rational function $\begin{bmatrix} m & n \\ k & l \end{bmatrix}_i$ satisfies the following recursive identity:

$$\begin{bmatrix} m & n \\ k & l \end{bmatrix}_i = \alpha_{m,n}^k \times \begin{bmatrix} m & n \\ k-1 & l-1 \end{bmatrix}_i + \beta_{m,n}^k \times \begin{bmatrix} m-1 & n-1 \\ k & l-1 \end{bmatrix}_{i-1}. \quad (4.3)$$

Proof. Substitute (4.1) in both sides of (3.5):

$$\begin{aligned} \sum_{i=0}^{\min(m,n,l)} \begin{bmatrix} m & n \\ k & l \end{bmatrix}_i \begin{array}{c} m \quad n \\ \text{---} \quad \text{---} \\ \text{ } i \text{ } \\ \text{ } k-l+i \text{ } \\ \text{---} \quad \text{---} \\ m' \quad n' \end{array} &= \alpha_{m,n}^k \sum_{i=0}^{\min(m,n,l-1)} \begin{bmatrix} m & n \\ k-1 & l-1 \end{bmatrix}_i \begin{array}{c} m \quad n \\ \text{---} \quad \text{---} \\ \text{ } i \text{ } \\ \text{ } k-l+i \text{ } \\ \text{---} \quad \text{---} \\ m' \quad n' \end{array} \\ &+ \beta_{m,n}^k \sum_{i=0}^{\min(m-1,n-1,l-1)} \begin{bmatrix} m-1 & n-1 \\ k & l-1 \end{bmatrix}_i \begin{array}{c} m \quad n \\ \text{---} \quad \text{---} \\ \text{ } 1 \text{ } \\ \text{ } i \text{ } \\ \text{ } k-l+i+1 \text{ } \\ \text{---} \quad \text{---} \\ m' \quad n' \end{array} \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=0}^{\min(m,n,l)} \begin{bmatrix} m & n \\ k & l \end{bmatrix}_i \begin{array}{c} m \quad n \\ \text{---} \quad \text{---} \\ \text{ } i \text{ } \\ \text{ } k-l+i \text{ } \\ \text{---} \quad \text{---} \\ m' \quad n' \end{array} &= \sum_{i=0}^{\min(m,n,l-1)} \alpha_{m,n}^k \times \begin{bmatrix} m & n \\ k-1 & l-1 \end{bmatrix}_i \begin{array}{c} m \quad n \\ \text{---} \quad \text{---} \\ \text{ } i \text{ } \\ \text{ } k-l+i \text{ } \\ \text{---} \quad \text{---} \\ m' \quad n' \end{array} \\ &+ \sum_{i=0}^{\min(m-1,n-1,l-1)} \beta_{m,n}^k \times \begin{bmatrix} m-1 & n-1 \\ k & l-1 \end{bmatrix}_i \begin{array}{c} m \quad n \\ \text{---} \quad \text{---} \\ \text{ } i+1 \text{ } \\ \text{ } k-l+i+1 \text{ } \\ \text{---} \quad \text{---} \\ m' \quad n' \end{array} \end{aligned}$$

The latter can be written as

$$\begin{aligned}
 \sum_{i=0}^{\min(m,n,l)} \left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i &= \sum_{i=0}^{\min(m,n,l-1)} \alpha_{m,n}^k \times \left[\begin{matrix} m & n \\ k-1 & l-1 \end{matrix} \right]_i \\
 &+ \sum_{i=1}^{\min(m,n,l)} \beta_{m,n}^k \times \left[\begin{matrix} m-1 & n-1 \\ k & l-1 \end{matrix} \right]_{i-1}
 \end{aligned}$$

Now note that the elements $\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i$, where $0 \leq i \leq \min(m, n, l)$, are linearly independent

in the module $\mathcal{D}_{m',n'}^{m,n}$. Hence we conclude that

$$\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i = \alpha_{m,n}^k \times \left[\begin{matrix} m & n \\ k-1 & l-1 \end{matrix} \right]_i + \beta_{m,n}^k \times \left[\begin{matrix} m-1 & n-1 \\ k & l-1 \end{matrix} \right]_{i-1}.$$

□

Remark 4.3. The coefficients $\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i$ behave like the binomial coefficients $\binom{l}{i}$ in the sense that $\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i = 0$ when $i < 0$ or $i > l$. Note also that the recursion formula for $\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i$, when one focuses the variables l and i , is analogous to the recursion formula of the binomial coefficients $\binom{l}{i}$.

Theorem 4.4. Let $m, n, k, l \geq 0$, and $k \geq l$ and $0 \leq i \leq \min(m, n, l)$. Then

$$\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i = \frac{\prod_{j=0}^{l-i-1} \Delta_{k-j-1} \prod_{s=0}^{i-1} \Delta_{n-s-1} \Delta_{m-s-1}}{\prod_{t=0}^{l-1} \Delta_{n+k-t-1} \Delta_{m+k-t-1}} \sum_{\substack{p_0 \leq p_1 \leq \dots \leq p_{l-i-1} \\ 0 \leq p_j \leq i}} \prod_{j=0}^{l-i-1} \Delta_{m-p_j+n-p_j+k-j} \quad (4.4)$$

Proof. This equation agrees with the recursion identity (4.3). □

The symmetry of the bubble skein element, or the previous formula for the coefficients $\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i$, implies immediately the following.

Proposition 4.5. Let $m, n, k, l \geq 0$. Let $k \geq l$. Then

$$\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i = \left[\begin{matrix} n & m \\ k & l \end{matrix} \right]_i \quad (4.5)$$

It is preferred sometimes to write the coefficients $\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i$ in terms of quantum integers rather than deltas. Recall that $\Delta_n = (-1)^n[n+1]$ and hence the sign of the term $\prod_{j=0}^{l-i-1} [k-j]$ can be easily calculated to be $-(\frac{1}{2})(i-l)(-1+i+2k-l)$. Similarly the sign of $\prod_{s=0}^{i-1} [n-s]$ is $-(\frac{1}{2})i(1+i-2n)$, the sign of $\prod_{s=0}^{i-1} [m-s]$ is $-(\frac{1}{2})i(1+i-2m)$, the sign of $\prod_{t=0}^{l-1} [n+k-t]$ is $-(\frac{1}{2})l(1-2k+l-2n)$, and the sign of $\prod_{j=0}^{l-i-1} [m-p_j+n-p_j+k-j+1]$ is $-(\frac{1}{2})(i-l)(1+i+2k-l+2m+2n)$. Thus the sign of the whole term is $-i-2i^2-2ik-l+2il+4kl-2l^2+2lm+2ln = i+l \pmod{2}$. Hence the previous theorem can be rewritten as follows:

Corollary 4.6. Let $m, n, k, l \geq 0$. Let $k \geq l$ and $0 \leq i \leq \min(m, n, l)$. Then

$$\left[\begin{matrix} m & n \\ k & l \end{matrix} \right]_i = (-1)^{i+l} \frac{\prod_{j=0}^{l-i-1} [k-j] \prod_{s=0}^{i-1} [n-s] [m-s]}{\prod_{t=0}^{l-1} [n+k-t] [m+k-t]} \sum_{\substack{p_0 \leq p_1 \leq \dots \leq p_{l-i-1} \\ 0 \leq p_j \leq i}} \prod_{j=0}^{l-i-1} [m-p_j+n-p_j+k-j+1]$$

5. APPLICATIONS

5.1. The theta graph. A theta graph is a spin network in S^2 that plays an important role in computing arbitrary spin network in S^2 . The evaluation of a theta graph in $\mathcal{S}(S^2)$ is equivalent to find the evaluation of the skein element in Figure 3. Explicit determination of this skein element was

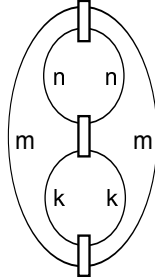


FIGURE 3.

done in [8] and [12]. Our first immediate application of the bubble expansion formula is computing this skein element. The following lemma shows that $\left[\begin{matrix} m & n \\ k & k \end{matrix} \right]_0$ is almost equal to the evaluation of this skein element.

Lemma 5.1.

$$\begin{array}{c} \text{Diagram: A large oval with two internal bubbles. The top bubble has two 'n' labels. The bottom bubble has two 'k' labels. The left and right sides of the oval are labeled 'm'. There are four small vertical bars on the oval boundary: one at the top, one between the bubbles, one at the bottom, and one on the right side. } \end{array} = \begin{bmatrix} m & n \\ k & k \end{bmatrix}_0 \Delta_{m+n} \quad (5.1)$$

Proof. We apply the bubble skein formula (4.1) on that bubble $\mathcal{B}_{m,n}^{m,n}(k,k)$ appears in Figure 3:

$$\begin{array}{c} \text{Diagram: Same as Lemma 5.1, but with a summation over i. The top bubble has 'n' and 'n-i' labels. The bottom bubble has 'k' and 'i' labels. The right side of the oval is labeled 'm'. There are four small vertical bars on the oval boundary: one at the top, one between the bubbles, one at the bottom, and one on the right side. } \end{array} = \sum_{i=0}^{\min(m,n,k)} \begin{bmatrix} m & n \\ k & k \end{bmatrix}_i \begin{array}{c} \text{Diagram: A large oval with two internal bubbles. The top bubble has two 'n' labels. The bottom bubble has two 'i' labels. The left and right sides of the oval are labeled 'm'. There are four small vertical bars on the oval boundary: one at the top, one between the bubbles, one at the bottom, and one on the right side. } \end{array}$$

The previous summation is zero except when $i = 0$ and hence it reduces to

$$\begin{array}{c} \text{Diagram: Same as Lemma 5.1, but with a summation over i. The top bubble has two 'n' labels. The bottom bubble has two 'k' labels. The left and right sides of the oval are labeled 'm'. There are four small vertical bars on the oval boundary: one at the top, one between the bubbles, one at the bottom, and one on the right side. } \end{array} = \begin{bmatrix} m & n \\ k & k \end{bmatrix}_0 \Delta_{m+n}$$

□

Remark 5.2. In the previous lemma one could apply the bubble skein formula on the other bubbles in the skein element 3 and obtain

$$\begin{array}{c} \text{Diagram: Same as Lemma 5.1, but with a summation over i. The top bubble has two 'n' labels. The bottom bubble has two 'k' labels. The left and right sides of the oval are labeled 'm'. There are four small vertical bars on the oval boundary: one at the top, one between the bubbles, one at the bottom, and one on the right side. } \end{array} = \begin{bmatrix} m & n \\ k & k \end{bmatrix}_0 \Delta_{m+n} = \begin{bmatrix} m & k \\ n & n \end{bmatrix}_0 \Delta_{m+k} = \begin{bmatrix} n & k \\ m & m \end{bmatrix}_0 \Delta_{n+k}.$$

This reflects the symmetry of the theta graph.

5.2. The head and the tail of alternating knots. We briefly review the basics of the head and the tail of the colored Jones polynomial. For more details see [1] and [2]. Then we use the bubble expansion formula to give an example of calculation that demonstrates how one can use the bubble expansion formula to study certain skein elements in $\mathcal{S}(S^3)$ that determines the tail of a family of knots.

We start by recalling the definition of the unreduced colored Jones polynomial for links in S^3 . Let L be a framed link in S^3 . Decorate every component of L , according to its framing, by the n^{th}

Jones-Wenzl idempotent and consider this decorated framed link as an element of $\mathcal{S}(S^3)$. Up to a power of $\pm A$, that depends on the framing of L , the value of this element is the n^{th} (unreduced) colored Jones polynomial $\tilde{J}_{n,L}(A)$. To recover the reduced colored Jones polynomial one makes the substitution $A = q^{-1/4}$ and divide $\tilde{J}_{n,L}(A)$ by the value of Δ_n . Hence

$$J_{n+1,L}(q) = \frac{\tilde{J}_{n,L}(A)}{\Delta_n} \Big|_{A=q^{-1/4}}.$$

Following [2], write $P_1(q) \doteq_n P_2(q)$, where $P_1(q), P_2(q)$ are two Laurent series if and only if $P_1(q) = q^{\pm s} P_2(q) \pmod{q^n}$. It was proven in [1] that k^{th} coefficient in the entire colored Jones polynomial of an alternating link L does not depend on the color n as long as $n \geq k$. Thus one could define a power series associated with the entire colored Jones polynomial whose n^{th} coefficient is the n^{th} coefficient of $J_{n,L}$. The tail of the colored Jones polynomial of a link L is defined to be a series $T_L(q)$, that satisfies $T_L(q) \doteq_N J_{N,L}(q)$ for all N . In the same way, the head of the colored Jones polynomial of a link L is defined to be the tail of $J_{N,L}(q^{-1})$. For more details see [1]. Higher order stability of the coefficients of the colored Jones polynomial were studied by Garoufalidis and Le in [3].

Let L be a link in S^3 and D be an alternating knot diagram of L . For any crossing in D there are two ways to smooth this crossing, the A -smoothing and the B -smoothing. See Figure 4.

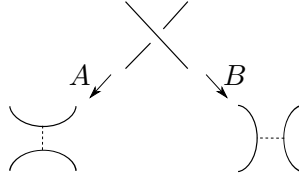
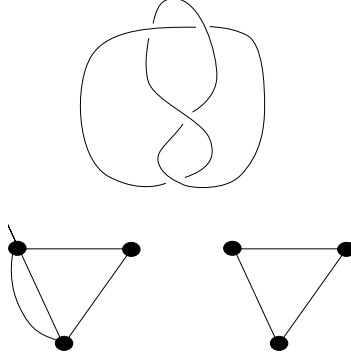


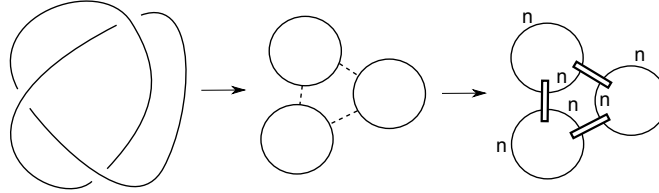
FIGURE 4. A and B smoothings

We replace a crossing with a smoothing together with a dashed line joining the two arcs. After applying a smoothing to each crossing in D we obtain diagram of a collection of disjoint circles in the plane. We call this diagram a *state* for the diagram D . The all- A smoothing state and the all- B smoothing for D are of particular importance for us. The all- A smoothing (all- B smoothing) state of D is the state obtained by replacing each crossing by a A smoothing (B smoothing). Write $S_A(D)$ and $S_B(D)$ to denote the all A smoothing and all B smoothing states of D respectively.

If D is a link diagram then the A -graph $A(D)$ and the B -graph $B(D)$ are two graphs associated to the all- A smoothing and all- B smoothing states of D . The set of vertices of $A(D)$ is equal to the set of circles in $S_A(D)$. Moreover, an edge in the set of edges of $A(D)$ is obtained by joining two vertices of $A(D)$ for each crossing in D between the corresponding circles. We obtain the *reduced* A -graph $A(D)'$ by keeping the same set of vertices of $A(D)$ and replacing parallel edges by a single edge. See Figure 5 for an example. We define the B -graph $B(D)$ and reduced B -graph $B(D)'$ similarly.

FIGURE 5. The knot 4_1 , its A -graph on the left and its reduced A -graph on the right

Now let D be a link diagram and consider the skein element obtained from $S_B(D)$ by decorating each circle in $S_B(D)$ with the n^{th} Jones-Wenzl idempotent and replacing each dashed line in $S_B(D)$ with the $(2n)^{\text{th}}$ Jones-Wenzl idempotent. See Figure 6 for an example. Write $S_B^{(n)}(D)$ to denote this skein element.

FIGURE 6. Obtaining $S_B^{(n)}(D)$ from a knot diagram D

We will need the following theorem.

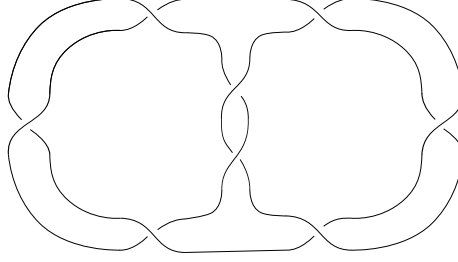
Theorem 5.3. (C. Armond [2]) Let L be a link in S^3 and D be a reduced alternating knot diagram of L . Then

$$\tilde{J}_{n,L} \doteq_{4(n+1)} S_B^{(n)}(D).$$

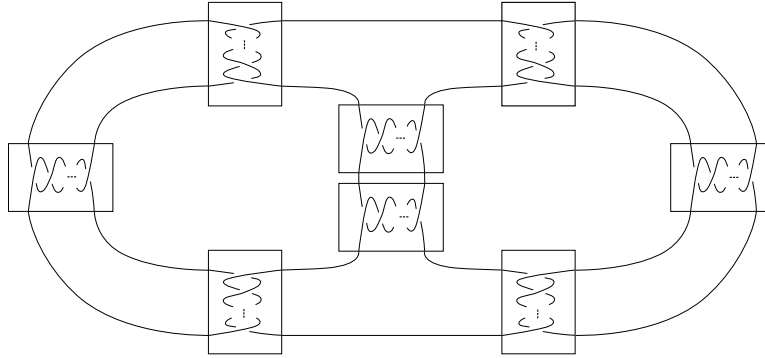
Theorem 5.3 has an important consequence, which basically tells us that the tail (the head) of an alternating link only depends on the reduced A -graph (reduced B -graph):

Theorem 5.4. (C. Armond, O. Dasbach [1]) Let L_1 and L_2 be two alternating links with alternating diagrams D_1 and D_2 . If the graph $A(D_1)'$ coincides with $A(D_2)'$, then $T_{K_1} = T_{K_2}$. Similarly, if $B(D_1)'$ coincides with $B(D_2)'$, then $H_{K_1} = H_{K_2}$.

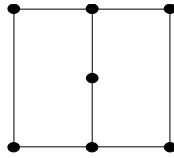
Finding an exact form for the head and tail series is an interesting task. Explicit calculations were done on the knot table to determine these two power series in [1]. Using multiple techniques Armond and Dasbach determined the head and tail for an infinite family of knots and links. The knot 8_5 , Figure 7, is the first knot on the knot table whose tail could not be determined by a direct application of techniques in [1].

FIGURE 7. The knot 8_5

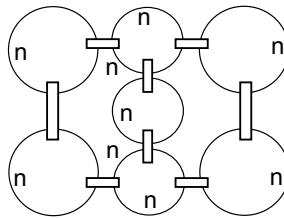
The techniques we developed here appear to be helpful in understanding the head and the tail for some knots. We demonstrate this by studying the family of knots in Figure 8. Note that we obtain the knot 8_5 from this family by replacing each crossing region by a single crossing.

FIGURE 8. The knot Γ

The reduced B -graph for each knot in this family can be easily seen to coincide with the graph in Figure 9.

FIGURE 9. The reduced B -graph for Γ

Hence the skein element $S_B^{(n)}$ for any knot from the family 8 is given in Figure 10. Note that the bubble skein element appears in multiple places in this skein element.

FIGURE 10. The skein element $S_B^{(n)}(\Gamma)$

Let Γ be any knot from the family 8. Theorem 5.3 implies

$$\tilde{J}_{n,\Gamma}(A) \doteq_{4(n+1)} \begin{array}{c} \text{Diagram with 6 bubbles arranged in a 2x3 grid. Each bubble is labeled } n. \text{ Horizontal and vertical connections between bubbles are labeled } n. \end{array} .$$

We use the bubble expansion formula on the top left bubble element:

$$\tilde{J}_{n,\Gamma}(A) \doteq_{4(n+1)} \sum_{i=0}^n \begin{bmatrix} n & n \\ n & n \end{bmatrix}_i \begin{array}{c} \text{Diagram where the top-left bubble is expanded into a sum of terms. The top part of the bubble is labeled } n-i \text{ and } i, \text{ and the bottom part is labeled } n-i \text{ and } n. \end{array}$$

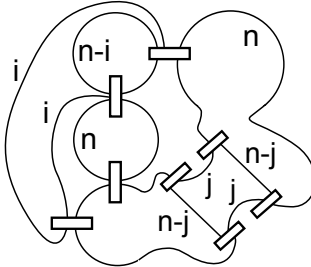
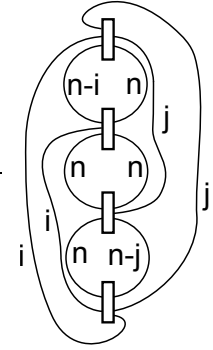
Using properties of the Jones-Wenzl projector

$$\tilde{J}_{n,\Gamma}(A) \doteq_{4(n+1)} \sum_{i=0}^n \begin{bmatrix} n & n \\ n & n \end{bmatrix}_i \begin{array}{c} \text{Diagram where the top-left bubble is expanded into a sum of terms. The top part of the bubble is labeled } n-i \text{ and } i, \text{ and the bottom part is labeled } n-i \text{ and } n. \end{array}$$

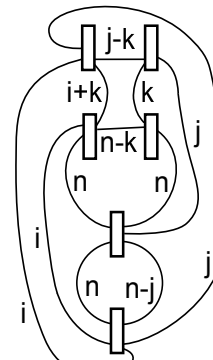
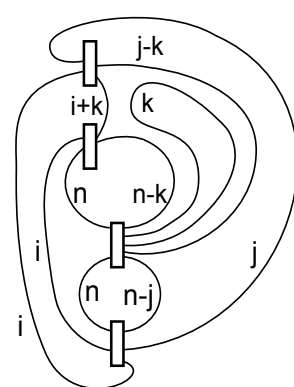
Using the identity (2.3)

$$\tilde{J}_{n,\Gamma}(A) \doteq_{4(n+1)} \sum_{i=0}^n \begin{bmatrix} n & n \\ n & n \end{bmatrix}_i \frac{\Delta_{2n}}{\Delta_{n+i}} \begin{array}{c} \text{Diagram where the top-left bubble is expanded into a sum of terms. The top part of the bubble is labeled } n-i \text{ and } i, \text{ and the bottom part is labeled } n-i \text{ and } n. \end{array} .$$

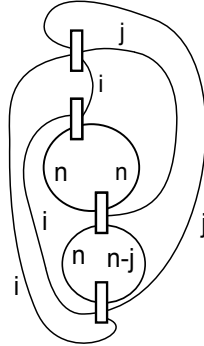
Now apply the bubble expansion formula to lower mostright bubble

$$\begin{aligned}
 \tilde{J}_{n,\Gamma}(A) &\stackrel{\doteq}{=}_{4(n+1)} \sum_{i=0}^n \sum_{j=0}^n \left[\begin{array}{cc} n & n \\ n & n \end{array} \right]_i \left[\begin{array}{cc} n & n \\ n & n \end{array} \right]_j \frac{\Delta_{2n}}{\Delta_{n+i}} \quad \text{Diagram 1} \\
 &= \sum_{i=0}^n \sum_{j=0}^n \left[\begin{array}{cc} n & n \\ n & n \end{array} \right]_i \left[\begin{array}{cc} n & n \\ n & n \end{array} \right]_j \frac{\Delta_{2n}}{\Delta_{n+i}} \frac{\Delta_{2n}}{\Delta_{n+j}} \quad \text{Diagram 2}
 \end{aligned}$$



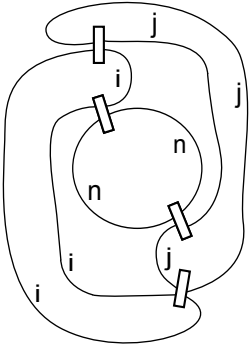
Similarly, we apply the bubble expansion formula on the top bubble that appears in the previous equation

$$\begin{aligned}
 \tilde{J}_{n,\Gamma}(A) &\stackrel{\doteq}{=}_{4(n+1)} \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^{\min(j, n-i)} \left[\begin{array}{cc} n & n \\ n & n \end{array} \right]_i \left[\begin{array}{cc} n & n \\ n & n \end{array} \right]_j \left[\begin{array}{cc} j & n \\ n & n-i \end{array} \right]_k \\
 &\times \frac{\Delta_{2n}}{\Delta_{n+i}} \frac{\Delta_{2n}}{\Delta_{n+j}} \quad \text{Diagram 3} \\
 &= \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^{\min(j, n-i)} \left[\begin{array}{cc} n & n \\ n & n \end{array} \right]_i \left[\begin{array}{cc} n & n \\ n & n \end{array} \right]_j \left[\begin{array}{cc} j & n \\ n & n-i \end{array} \right]_k \\
 &\times \frac{\Delta_{2n}}{\Delta_{n+i}} \frac{\Delta_{2n}}{\Delta_{n+j}} \quad \text{Diagram 4}
 \end{aligned}$$



Note that the previous sum is zero unless $k = 0$. Hence

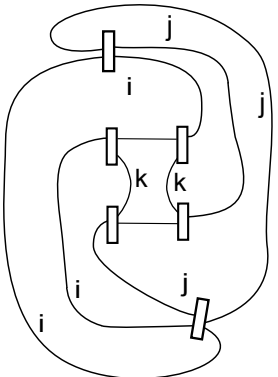
$$\tilde{J}_{n,\Gamma}(A) \doteq_{4(n+1)} \sum_{i=0}^n \sum_{j=0}^n \begin{bmatrix} n & n \\ n & n \end{bmatrix}_i \begin{bmatrix} n & n \\ n & n \end{bmatrix}_j \begin{bmatrix} j & n \\ n & n-i \end{bmatrix}_0 \frac{\Delta_{2n}}{\Delta_{n+i}} \frac{\Delta_{2n}}{\Delta_{n+j}}$$


Similarly

$$\tilde{J}_{n,\Gamma}(A) \doteq_{4(n+1)} \sum_{i=0}^n \sum_{j=0}^n \begin{bmatrix} n & n \\ n & n \end{bmatrix}_i \begin{bmatrix} n & n \\ n & n \end{bmatrix}_j \begin{bmatrix} i & n \\ n & n-j \end{bmatrix}_0 \begin{bmatrix} j & n \\ n & n-i \end{bmatrix}_0 \frac{\Delta_{2n}}{\Delta_{n+i}} \frac{\Delta_{2n}}{\Delta_{n+j}}$$


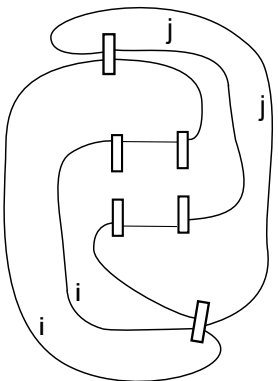
Applying the bubble expansion one last time on the only bubble left in previous summation, we obtain

$$\tilde{J}_{n,\Gamma}(A) \doteq_{4(n+1)} \sum_{i=0}^n \sum_{j=0}^n \sum_{k=0}^{\min(i,j)} \begin{bmatrix} n & n \\ n & n \end{bmatrix}_i \begin{bmatrix} n & n \\ n & n \end{bmatrix}_j \begin{bmatrix} i & n \\ n & n-j \end{bmatrix}_0 \begin{bmatrix} j & n \\ n & n-i \end{bmatrix}_0 \begin{bmatrix} j & i \\ n & n \end{bmatrix}_k$$

$$\times \frac{\Delta_{2n}}{\Delta_{n+i}} \frac{\Delta_{2n}}{\Delta_{n+j}}$$


The previous summation is zero unless $k = 0$. Hence

$$\tilde{J}_{n,\Gamma}(A) \doteq_{4(n+1)} \sum_{i=0}^n \sum_{j=0}^n \begin{bmatrix} n & n \\ n & n \end{bmatrix}_i \begin{bmatrix} n & n \\ n & n \end{bmatrix}_j \begin{bmatrix} i & n \\ n & n-j \end{bmatrix}_0 \begin{bmatrix} j & n \\ n & n-i \end{bmatrix}_0 \begin{bmatrix} j & i \\ n & n \end{bmatrix}_0$$

$$\times \frac{\Delta_{2n}}{\Delta_{n+i}} \frac{\Delta_{2n}}{\Delta_{n+j}}$$


which implies that

$$\tilde{J}_{n,\Gamma}(A) \doteq_{4(n+1)} \sum_{i=0}^n \sum_{j=0}^n \begin{bmatrix} n & n \\ n & n \end{bmatrix}_i \begin{bmatrix} n & n \\ n & n \end{bmatrix}_j \begin{bmatrix} i & n \\ n & n-j \end{bmatrix}_0 \begin{bmatrix} j & n \\ n & n-i \end{bmatrix}_0 \begin{bmatrix} j & i \\ n & n \end{bmatrix}_0 \frac{\Delta_{2n}}{\Delta_{n+i}} \frac{\Delta_{2n}}{\Delta_{n+j}} \Delta_{i+j}.$$

Using Mathematica we computed the first 120 terms of $T_{85}(q)$:

$$\begin{aligned} T_{85} \doteq_{120} & 1 - 2q + q^2 - 2q^4 + 3q^5 - 3q^8 + q^9 + 4q^{10} - q^{11} - 2q^{12} - 2q^{13} - 3q^{14} + 3q^{15} + 7q^{16} + 2q^{17} - 4q^{18} - \\ & 4q^{19} - 4q^{20} - 5q^{21} + 3q^{22} + 9q^{23} + 9q^{24} - 4q^{26} - 9q^{27} - 8q^{28} - 5q^{29} - q^{30} + 9q^{31} + 13q^{32} + 16q^{33} + 5q^{34} - \\ & 10q^{35} - 13q^{36} - 15q^{37} - 12q^{38} - 7q^{39} + 15q^{41} + 25q^{42} + 23q^{43} + 15q^{44} - 3q^{45} - 16q^{46} - 28q^{47} - 31q^{48} - \\ & 21q^{49} - 12q^{50} + 4q^{51} + 16q^{52} + 37q^{53} + 41q^{54} + 39q^{55} + 26q^{56} - 6q^{57} - 34q^{58} - 48q^{59} - 51q^{60} - 49q^{61} - \\ & 32q^{62} - 8q^{63} + 20q^{64} + 39q^{65} + 67q^{66} + 76q^{67} + 67q^{68} + 43q^{69} + 9q^{70} - 36q^{71} - 74q^{72} - 99q^{73} - 101q^{74} - \\ & 79q^{75} - 52q^{76} - 7q^{77} + 33q^{78} + 77q^{79} + 108q^{80} + 135q^{81} + 127q^{82} + 104q^{83} + 51q^{84} - 10q^{85} - 82q^{86} - \\ & 145q^{87} - 174q^{88} - 182q^{89} - 160q^{90} - 115q^{91} - 37q^{92} + 37q^{93} + 119q^{94} + 177q^{95} + 218q^{96} + 238q^{97} + 229q^{98} + \\ & 171q^{99} + 88q^{100} - 17q^{101} - 126q^{102} - 236q^{103} - 313q^{104} - 344q^{105} - 325q^{106} - 256q^{107} - 157q^{108} - 28q^{109} + \\ & 98q^{110} + 241q^{111} + 343q^{112} + 420q^{113} + 440q^{114} + 424q^{115} + 336q^{116} + 212q^{117} + 41q^{118} - 150q^{119} - 324q^{120}. \end{aligned}$$

REFERENCES

- [1] C. Armond, and O. T. Dasbach, *Rogers-Ramanujan Type Identities and the Head and Tail of the Colored Jones Polynomial*, 2011, arXiv:1106.3948v1.
- [2] C. Armond, *The head and tail conjecture for alternating knots*, 2011, arXiv:1112.3995v1.
- [3] S. Garoufalidis and T. T. Q. Le, *Nahm sums, stability and the colored Jones polynomial*, 2011, arXiv:1112.3905, Preprint.
- [4] M. Hajij, *In preparation*.
- [5] V. F. R. Jones, *Index of subfactors*, Invent. Math., 72 (1983), 1-25.
- [6] L. H. Kauffman, *State models and the Jones polynomial*, Topology, 26(1987), 395-401.
- [7] L. H. Kauffman, *An invariant of regular isotopy*, Transactions of the American Mathematical Society 318(2), 1990, pp. 317-371.
- [8] L. Kauffman, S. Lins, *Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds*, Princeton Univ. Press, 1994.
- [9] W.B.R. Lickorish, *3-manifolds and the Temperley-Lieb algebra*, Math. Annal, 290:657-670, 1991.
- [10] W.B.R. Lickorish, *The skein method for 3-manifold invariants*, J. Knot Theor. Ramif., 2 (1993), 171-194.
- [11] W.B.R. Lickorish, *An Introduction to Knot Theory*, Springer, 1997.
- [12] G. Masbaum, and P. Vogel, *3-valent graphs and the Kauffman bracket*, Pacific J. Math., 164 (1994), No. 2, 361-381.
- [13] G. Masbaum, *Skein-theoretical derivation of some formulas of Habiro*, Algebraic and Geometric Topology, 3 (2003), 537556.
- [14] J.H. Przytycki, *Fundamentals of Kauffman bracket skein module*, Kobe J. Math. 16 (1999), no. 1, 45-66.

- [15] J.H. Przytycki, *Skein modules of 3-manifolds*, Bull. Pol. Acad. Sci. 39(1-2) (1991) 91100.
- [16] N. Y. Reshetikhin and V. Turaev, *Invariants of three manifolds via link polynomials and quantum groups*, Invent. Math., 103:547-597, 1991.
- [17] H. Wenzl, *On sequences of projections*, C. R. Math. Rep. Acad. Sci. Canada, IX (1987), 5-9.

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803 USA
E-mail address: `mhajij1@math.lsu.edu`